

Linear rigidity of stationary stochastic processes

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Abstract

We consider stationary stochastic processes $\{X_n : n \in \mathbb{Z}\}$ such that X_0 lies in the closed linear span of $\{X_n : n \neq 0\}$; following Ghosh and Peres, we call such processes linearly rigid. Using a criterion of Kolmogorov, we show that it suffices, for a stationary stochastic process to be linearly rigid, that the spectral density vanish at zero and belong to the Zygmund class $\Lambda_*(1)$. We next give sufficient condition for stationary determinantal point processes on \mathbb{Z} and on \mathbb{R} to be linearly rigid. Finally, we show that the determinantal point process on \mathbb{R}^2 induced by a tensor square of Dyson sine-kernels is *not* linearly rigid.

Keywords. Stationary stochastic processes, the Kolmogorov criterion, stationary determinantal point processes, rigidity

1 Introduction

This paper is devoted to rigidity of stationary determinantal point processes.

Recall that stationary determinantal point processes are strongly chaotic: they have the Kolmogorov property (Lyons [11]) and the Bernoulli property (Lyons and Steif [12]); and they satisfy the Central Limit Theorem (Costin and Lebowitz [2], Soshnikov [16]). On the other hand, Ghosh [5] and Ghosh-Peres [6] proved, for the determinantal point processes such as Dyson sine process and Ginibre point process, that number of particles in a finite

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window is measurable with respect to the completion of the sigma-algebra describing the configurations outside that finite window. Their argument is spectral: they construct, for any small ε , a compactly supported smooth function φ_ε , such that φ_ε equals 1 in a fixed finite window and the linear statistic corresponding to φ_ε has variance smaller than ε .

In the same spirit, we consider general stationary stochastic processes (in broad sense) $\{X_n : n \in \mathbb{Z}\}$ such that X_0 lies in the closed linear span of X_n , $n \neq 0$; following Ghosh and Peres, we call such processes linearly rigid. In 1941 Kolmogorov [9], [10] gave a sufficient condition for linear rigidity: namely, that the spectral density of our process vanish at zero and the integral of the inverse of the spectral density diverge. Such a condition is easy to verify for example for the sine-process, since the spectral density ω in the neighbourhood of zero has the form $\omega(\theta) = |\theta|$. More generally, in order that a stationary stochastic process be rigid, we check that it suffices that the spectral density vanish at zero and belong to the Zygmund class $\Lambda_*(1)$. We next give sufficient condition for stationary determinantal point processes on \mathbb{Z} and on \mathbb{R} to be rigid. Finally, we show that the determinantal point process on \mathbb{R}^2 induced by a tensor square of Dyson sine-kernel is *not* linearly rigid.

We now turn to more precise statements. Let $X = \{X_n : n \in \mathbb{Z}^d\}$ be a multi-dimensional time stationary stochastic process of real-valued random variables defined on a probability space (Ω, \mathbb{P}) . Let $H(X) \subset L^2(\Omega, \mathbb{P})$ denote the closed subspace linearly spanned by $\{X_n : n \in \mathbb{Z}^d\}$ and let $\check{H}_0(X)$ denote the one linearly spanned by $\{X_n : n \in \mathbb{Z}^d \setminus \{0\}\}$.

Definition 1.1. The stochastic process X is said to be linearly rigid if

$$X_0 \in \check{H}_0(X). \quad (1)$$

Let $\text{Conf}(\mathbb{R}^d)$ be the set of locally finite configurations on \mathbb{R}^d . For a bounded Borel subset $B \subset \mathbb{R}^d$, we denote $N_B : \text{Conf}(\mathbb{R}^d) \rightarrow \mathbb{N} \cup \{0\}$ the function defined by

$$N_B(\mathcal{X}) := \text{the cardinality of } B \cap \mathcal{X}.$$

The space $\text{Conf}(\mathbb{R}^d)$ is equipped with the Borel σ -algebra which is the smallest σ -algebra making all N_B 's measurable. Recall that a point process with phase space \mathbb{R}^d is, by definition, a Borel probability measure on the space $\text{Conf}(\mathbb{R}^d)$. For the background on point process, the reader is referred to Daley and Vere-Jones' book [3].

Given a stationary point process on \mathbb{R}^d and $\lambda > 0$, we introduce the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ by the formula

$$N_n^{(\lambda)}(\mathcal{X}) := \text{the cardinality of } \mathcal{X} \cap (n\lambda + [-\lambda/2, \lambda/2)^d). \quad (2)$$

Definition 1.2. A stationary point process \mathbb{P} on \mathbb{R}^d is called **linearly rigid**, if for any $\lambda > 0$, the stationary stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ is linearly rigid, i.e.,

$$N_0^{(\lambda)} \in \check{H}_0(N^{(\lambda)}).$$

The above definition is motivated by the definition due to Ghosh and Peres of rigidity of point processes on \mathbb{R}^d , see [5] and [6]. Given a Borel subset $C \subset \mathbb{R}^d$, we will denote

$$\mathcal{F}_C = \sigma(\{N_B : B \subset C, B \text{ bounded Borel}\})$$

the σ -algebra generated by all random variables of the form N_B where $B \subset C$ ranges over all bounded Borel subsets of C . Let \mathbb{P} be a point process on \mathbb{R} , i.e., \mathbb{P} is a Borel probability on $\text{Conf}(\mathbb{R}^d)$, and denote $\mathcal{F}_C^{\mathbb{P}}$ for the \mathbb{P} -completion of \mathcal{F}_C .

Definition 1.3 (Ghosh [5], Ghosh-Peres [6]). A point process \mathbb{P} on \mathbb{R}^d is called **number rigid**, if for any bounded Borel set $B \subset \mathbb{R}^d$ with Lebesgue-negligible boundary ∂B , the random variable N_B is $\mathcal{F}_{\mathbb{R}^d \setminus B}^{\mathbb{P}}$ -measurable.

Remark 1.1. Of course, in the above definition, it suffices to take Borel sets B of the form $[-\gamma, \gamma]^d$ for $\gamma > 0$, cf. [6].

A linear rigid stationary point process on \mathbb{R}^d is of course rigid in the sense of Ghosh and Peres. Observe that proofs for rigidity in [5], [6] and [1] in fact establish linear rigidity. We would like also to mention a notion of insertion-deletion tolerance studied by Holroyd and Soo in [7], which is in contrast to the notion of rigidity property.

2 The Kolmogorov criterion for linear rigidity

In this note, the Fourier transform of a function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i2\pi x \cdot \xi} dx.$$

Denote by $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ the d -dimensional torus. In what follows, we identify \mathbb{T}^d with $[-1/2, 1/2)^d$. The Fourier coefficients of a measure μ on \mathbb{T}^d are given, for any $k \in \mathbb{Z}^d$, by the formula

$$\hat{\mu}(k) = \int_{\mathbb{T}^d} e^{-i2\pi k \cdot \theta} d\mu_X(\theta), \text{ where } k \cdot \theta := k_1\theta_1 + \cdots + k_d\theta_d.$$

Denote by μ_X the spectral measure of X , i.e.,

$$\forall k \in \mathbb{Z}^d, \quad \mathbb{E}(X_0 X_k) = \mathbb{E}(X_n X_{n+k}) = \int_{\mathbb{T}^d} e^{-i2\pi k \cdot \theta} d\mu_X(\theta) = \hat{\mu}_X(k). \quad (3)$$

Recall that we have the following natural isometric isomorphism

$$H(X) \simeq L^2(\mathbb{T}^d, \mu_X), \quad (4)$$

by assigning to $X_n \in H(X)$ the function $\theta \mapsto e^{i2\pi n \cdot \theta} \in L^2(\mathbb{T}^d, \mu_X)$.

Let $\mu_X = \mu_a + \mu_s$ be the Lebesgue decomposition of μ_X with respect to the normalized Lebesgue measure $m(d\theta) = d\theta_1 \cdots d\theta_d$ on \mathbb{T}^d , i.e., μ_a is absolutely continuous with respect to m and μ_s is singular to m . Set

$$\omega_X(\theta) := \frac{d\mu_a}{dm}(\theta).$$

Lemma 2.1 (The Kolmogorov Criterion). *We have*

$$\text{dist}(X_0, \check{H}_0(X)) = \left(\int_{\mathbb{T}^d} \omega_X^{-1} dm \right)^{-1/2},$$

where by $\text{dist}(X_0, \check{H}_0(X))$ we mean the least L^2 -distance between the random variable X_0 and the linear space $\check{H}_0(X)$ and the right side is to be interpreted as zero if $\int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty$.

Corollary 2.2. *The stationary stochastic process $X = (X_n)_{n \in \mathbb{Z}^d}$ is linearly rigid if and only if*

$$\int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty.$$

Lemma 2.1 is due to Kolmogorov [9], [10]. For the reader's convenience, we include its proof.

Proof of Lemma 2.1. We follow the argument of Lyons-Steif [12]. By the Lebesgue decomposition of μ , we may take a subset $A \subset \mathbb{T}^d$ of full Lebesgue measure $m(A) = 1$, such that $\mu_a(A) = 1$ and $\mu_s(A) = 0$.

Denote

$$L_0 = \overline{\text{span}}^{L^2(\mathbb{T}^d, \mu_X)}[e^{i2\pi n \cdot \theta} : n \neq 0].$$

By the isometric isomorphism (4), it suffices to show that

$$\text{dist}(1, L_0) = \left(\int_{\mathbb{T}^d} \omega_X^{-1} dm \right)^{-1/2}, \quad (5)$$

where 1 is the constant function taking value 1. Write

$$1 = p + h, \quad \text{such that } p \perp L_0, h \in L_0.$$

Modifying, if necessary, the values of p and h on a μ -negligible subset, we may assume that

$$1 = p(\theta) + h(\theta) \quad \text{for all } \theta \in \mathbb{T}^d.$$

Since $p \perp L_0$, we have

$$0 = \langle p, e^{i2\pi n \cdot \theta} \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} p(\theta) e^{-i2\pi n \cdot \theta} d\mu(\theta), \text{ for any } n \in \mathbb{Z}^d \setminus \{0\}. \quad (6)$$

Let $\xi \in \mathbb{C}$ denote

$$\xi = \int_{\mathbb{T}^d} p(\theta) d\mu(\theta).$$

Then by (6), all the Fourier coefficients of the complex measure $p \cdot d\mu$ coincide with the corresponding Fourier coefficients of ξdm (the multiple of Lebesgue measure dm by ξ), consequently, we have

$$p \cdot d\mu = \xi dm.$$

It follows that p must vanish almost everywhere with respect to the singular component μ_s of μ , and $p(\theta)\omega_X(\theta) = \xi$ for m -almost every $\theta \in \mathbb{T}^d$. Thus we have

$$\|p\|_{L^2(d\mu)} = \|p\|_{L^2(d\mu_a)}, \quad (7)$$

and

$$h(\theta) = 1 - \xi\omega_X(\theta)^{-1} \text{ for } m\text{-almost every } \theta \in \mathbb{T}^d. \quad (8)$$

Case 1: $\int_{\mathbb{T}^d} \omega_X^{-1} dm < \infty$.

Define a function $f : \mathbb{T}^d \rightarrow \mathbb{C}$ by $f = \omega_X^{-1} \chi_A$. Then $f \in L^2(d\mu) \ominus L_0$. Indeed,

$$\|f\|_{L^2(d\mu)}^2 = \int_{\mathbb{T}^d} \omega_X^{-2} \chi_A d\mu = \int_{\mathbb{T}^d} \omega_X^{-2} d\mu_a = \int_{\mathbb{T}^d} \omega_X^{-1} dm < \infty.$$

And, for all $n \in \mathbb{Z}^d \setminus 0$,

$$\langle f, e^{i2\pi n \cdot \theta} \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} \omega_X(\theta)^{-1} \chi_A(\theta) e^{-i2\pi n \cdot \theta} d\mu(\theta) = \int_{\mathbb{T}^d} e^{-i2\pi n \cdot \theta} dm(\theta) = 0.$$

It follows that $f \perp h$, i.e.,

$$0 = \langle h, f \rangle_{L^2(d\mu)} = \int_{\mathbb{T}^d} h \omega_X^{-1} \chi_A d\mu = \int_{\mathbb{T}^d} h dm.$$

By (8), we get

$$\int_{\mathbb{T}^d} (1 - \xi\omega_X^{-1}) dm = 0,$$

and hence

$$\xi = \left(\int_{\mathbb{T}^d} \omega_X^{-1} dm \right)^{-1}.$$

It follows that

$$\text{dist}(1, L_0)^2 = \|p\|_{L^2(d\mu)}^2 = \|p\|_{L^2(d\mu_a)}^2 = \xi^2 \int_{\mathbb{T}^d} \omega_X^{-2} \omega_X dm = \xi.$$

This shows the desired equality (5).

Case 2: $\int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty$.

We claim that $\xi = 0$. If the claim were verified, then we would get the desired identity in this case

$$\text{dist}(1, L_0) = 0.$$

So let us turn to the proof of the claim. We argue by contradiction. If $\xi \neq 0$, then $p \neq 0$ and

$$\|p\|_{L^2(d\mu)}^2 = \|p\|_{L^2(d\mu_a)}^2 = \xi^2 \|\omega_X^{-1}\|_{L^2(d\mu_a)}^2 = \xi^2 \int_{\mathbb{T}^d} \omega_X^{-1} dm = \infty.$$

This contradicts the fact that $p \in L^2(d\mu)$. \square

Remark 2.1. If the spectral measure μ_X is absolutely continuous and given by $\mu_X(dz) = \omega(z) dm(z)$, then for any $n \in \mathbb{N}$, the following are equivalent:

$$(i) \quad \sum_{l=-n}^n X_l \in \overline{\text{span}}^{H(X)} \{X_k : k \in \mathbb{Z}, |k| \geq n+1\}.$$

$$(i)' \quad \sum_{l=-n}^n z^l \in \overline{\text{span}}^{L_\omega^2} \{z^k : k \in \mathbb{Z}, |k| \geq n+1\}.$$

$$(ii) \quad \text{For any } w_1, w_2, \dots, w_n \in \mathbb{C} \setminus \{1\},$$

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^n |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)} dm(z) = \infty.$$

$$(ii)' \quad \text{For any } w_1, w_2, \dots, w_n \in \mathbb{T} \setminus \{1\},$$

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^n |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)} dm(z) = \infty.$$

Indeed, (i) and (i)' are equivalent. Assume (i)' is satisfied, let us show (ii). If (ii) is violated, then there exist $w_1, w_2, \dots, w_n \in \mathbb{C} \setminus \{1\}$, such that

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^n |(z - w_l)(z - \bar{w}_l)|^2}{\omega_X(z)} dm(z) < \infty.$$

Define

$$h(z) := \frac{\prod_{l=1}^n (z - w_l)(z - \bar{w}_l)}{z^n \omega_X(z)} = \frac{\sum_{l=-n}^n a_l z^l}{\omega_X(z)}.$$

Then $h \in L_\omega^2(\mathbb{T}) \ominus \overline{\text{span}}^{L_\omega^2} \{z^k : k \in \mathbb{Z}, |k| \geq n+1\}$. We have

$$\left(\sum_{l=-n}^n z^l, h(z) \right)_{L_\omega^2} = \sum_{l=-n}^n a_l = \prod_{l=1}^n |1 - w_l|^2 \neq 0.$$

This contradicts (i)', hence (i)' implies (ii).

Conversely, let us assume (ii) and show (i)'. If (i)' is not satisfied, then there exists a function $g \in L_\omega^2 \ominus \overline{\text{span}}^{L_\omega^2} \{z^l : k \in \mathbb{Z}, |k| \geq n+1\}$, such that $g \neq 0$ and the scalar product $(\sum_{l=-n}^n z^l, g)_{L_\omega^2} \neq 0$. We have

$$0 = \int_{\mathbb{T}} g(z) z^k \omega(z) dm(z), \text{ for any } k \in \mathbb{Z}, |k| \geq n+1.$$

This implies that there exists (c_{-n}, \dots, c_n) such that

$$g(z)\omega(z) = \sum_{-n}^n c_l z^l.$$

Hence $g(z) = \frac{\sum_{-n}^n c_l z^l}{\omega(z)}$ and

$$\sum_{-n}^n c_l = \left(g, \sum_{-n}^n z^l \right)_{L_\omega^2} \neq 0.$$

Since $\omega(z) = \omega(z^{-1})$, if we denote $\check{g}(z) := g(z^{-1})$, then $\check{g} \in L_\omega^2(\mathbb{T})$. Thus $\Re(g + \check{g})$ and $\Im(g + \check{g})$ are functions in $L_\omega^2(\mathbb{T})$. We have

$$\Re(g + \check{g})(z) = \frac{\sum_{-n}^n \Re(c_l)(z^l + z^{-l})}{\omega(z)} \quad \text{and} \quad \Im(g + \check{g})(z) = \frac{\sum_{-n}^n \Im(c_l)(z^l + z^{-l})}{\omega(z)}.$$

Since $\sum_{-n}^n c_l \neq 0$, we may assume without loss of generality that $\sum_{-n}^n \Re(c_l) \neq 0$. Define $P(z)$ the polynomial given by $P(z) = z^n \sum_{-n}^n \Re(c_l)(z^l + z^{-l})$ and let $m = \deg P \leq n$ then there exist w_1, \dots, w_m such that

$$P(z) = \Re(c_m) \prod_{l=1}^m (z - w_l)(z - \bar{w}_l).$$

Since $P(1) = \sum_{-n}^n \Re(c_l) \neq 0$, we know that w_1, \dots, w_m are all different from 1. Now using the fact $\Re(g + \check{g}) \in L_\omega^2$, we deduce that

$$\int_{\mathbb{T}} \frac{\prod_{l=1}^m |(z - w_l)(z - \bar{w}_l)|^2}{\omega(z)} < \infty,$$

which of course violates (ii). This contradiction shows that (ii) implies (i)'.

The equivalence between (ii) and (ii)' is obvious.

Denote by $\text{Cov}(U, V)$ the covariance between two random variables U and V : $\text{Cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)$.

If $X = (X_n)_{n \in \mathbb{Z}^d}$ is a stochastic process such that

$$\sum_{n \in \mathbb{Z}^d} |\text{Cov}(X_0, X_n)| < \infty, \tag{9}$$

then we may define a continuous function on \mathbb{T}^d by the formula

$$\omega_X(\theta) := \sum_{n \in \mathbb{Z}^d} \text{Cov}(X_0, X_n) e^{i2\pi n \cdot \theta}. \tag{10}$$

Lemma 2.3. *Let $X = (X_n)_{n \in \mathbb{Z}^d}$ be a stationary stochastic process satisfying condition (9). Then we have the following explicit Lebesgue decomposition of μ_X :*

$$\mu_X = (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m, \quad (11)$$

where δ_0 is the Dirac measure on the point $0 \in \mathbb{T}^d$ and ω_X is the function on \mathbb{T}^d defined by (10).

Proof. Note that, under the assumption (9), the function $\omega_X(\theta)$ is well-defined and continuous on \mathbb{T}^d . For proving the decomposition (11), it suffices to show that the Fourier coefficients of μ_X coincide with those of $\nu_X := (\mathbb{E}X_0)^2 \cdot \delta_0 + \omega_X \cdot m$. But if $n \in \mathbb{Z}^d$, then

$$\hat{\nu}_X(n) = (\mathbb{E}X_0)^2 + \text{Cov}(X_0, X_n) = \mathbb{E}(X_0 X_n) = \hat{\mu}_X(n).$$

The lemma is completely proved. \square

3 A sufficient condition for linear rigidity

Theorem 3.1. *Let $X = (X_n)_{n \in \mathbb{Z}}$ be a stationary stochastic process. If*

$$\sup_{N \geq 1} \left(N \sum_{|n| \geq N} |\text{Cov}(X_0, X_n)| \right) < \infty, \quad (12)$$

and

$$\sum_{n \in \mathbb{Z}} \text{Cov}(X_0, X_n) = 0. \quad (13)$$

Then X is linearly rigid.

Remark 3.1. The condition (12) is a sufficient condition such that the spectral density ω_X is a function in the Zygmund class $\Lambda_*(1)$, see below for definition. The condition (13) implies in particular that ω_X vanishes at the point $0 \in \mathbb{T}$.

We shall apply a result of F. Móricz [14, Thm. 3] on absolutely convergent Fourier series and Zygmund class functions. Recall that a continuous 1-periodic function φ defined on \mathbb{R} is said to be in the Zygmund class $\Lambda_*(1)$, if there exists a constant C such that

$$|\varphi(x+h) - 2\varphi(x) + \varphi(x-h)| \leq Ch \quad (14)$$

for all $x \in \mathbb{R}$ and for all $h > 0$.

Theorem 3.2 (Móricz, [14]). *If $\{c_n\}_{n \in \mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N \geq 1} \left(N \sum_{|n| \geq N} |c_n| \right) < \infty, \quad (15)$$

then the function $\varphi(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{i2\pi n\theta}$ is in the Zygmund class $\Lambda_*(1)$.

Proof of Theorem 3.1. First, in view of (10), our assumption (13) implies

$$\omega_X(0) = 0.$$

Next, by Theorem 3.2, under the assumption (12), we have

$$\omega_X \in \Lambda_*(1).$$

Since all Fourier coefficients of ω_X are real, we have

$$\omega_X(\theta) = \omega_X(-\theta).$$

Consequently, there exists $C > 0$, such that

$$\omega_X(\theta) = \frac{\omega_X(\theta) + \omega_X(-\theta)}{2} = \frac{\omega_X(\theta) + \omega_X(-\theta) - 2\omega_X(0)}{2} \leq C|\theta|,$$

whence

$$\int_{\mathbb{T}} \omega_X^{-1} dm = \infty,$$

and the stochastic process $X = (X_n)_{n \in \mathbb{Z}}$ is linearly rigid by the Kolmogorov criterion. \square

4 Applications to stationary determinantal point processes

In this section, we first give a sufficient condition for linear rigidity of stationary determinantal point processes on \mathbb{R} and then give an example of a very simple stationary, but not linearly rigid, determinantal point process on \mathbb{R}^2 . We briefly recall the main definitions. Let $B \subset \mathbb{R}^d$ be a bounded Borel subset. Let $K_B : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be the operator of convolution with the Fourier transform $\widehat{\chi_B}$ of the indicator function χ_B . In other words, the kernel of K_B is

$$K_B(x, y) = \widehat{\chi_B}(x - y). \quad (16)$$

In particular, if $d = 1$ and $B = (-1/2, 1/2)$, then we find the well-known Dyson sine kernel

$$K_{\text{sine}}(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

Note that we always have $K_B(x, x) = K_B(0, 0)$.

Denote by \mathbb{P}_{K_B} the determinantal point process induced by K_B . For the background on the determinantal point processes, the reader is referred to [8], [11], [13], [16].

Proposition 4.1. *Let \mathbb{P}_{K_B} be the stationary determinantal point process on \mathbb{R}^d induced by the kernel K_B in (16). For any $\lambda > 0$, denote by $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ the stationary stochastic process associated to \mathbb{P}_{K_B} as in (2). Then*

$$\sum_{n \in \mathbb{Z}^d} |\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| < \infty \quad (17)$$

and

$$\sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \quad (18)$$

Proof. Fix a number $\lambda > 0$, for simplifying the notation, let us denote $N_n^{(\lambda)}$ by N_n . Denote for any $n \in \mathbb{Z}^d$,

$$Q_n = n\lambda + [-\lambda/2, \lambda/2]^d.$$

By definition of a determinantal point process, we have

$$\mathbb{E}(N_n) = \mathbb{E}(N_0) = \int_{Q_0} K_B(x, x) dx = \lambda^d K_B(0, 0).$$

If $n \neq 0$, we have

$$\begin{aligned} \mathbb{E}(N_0 N_n) &= \iint \chi_{Q_0}(x) \chi_{Q_n}(y) \begin{vmatrix} K_B(x, x) & K_B(x, y) \\ K_B(y, x) & K_B(y, y) \end{vmatrix} dx dy \\ &= \lambda^{2d} K_B(0, 0)^2 - \iint_{Q_0 \times Q_n} |K_B(x, y)|^2 dx dy, \end{aligned}$$

whence

$$\text{Cov}(N_0, N_n) = - \iint_{Q_0 \times Q_n} |K_B(x, y)|^2 dx dy. \quad (19)$$

We also have

$$\begin{aligned} \mathbb{E}(N_0^2) &= \mathbb{E} \left[\sum_{x, y \in \mathcal{X}} \chi_{Q_0}(x) \chi_{Q_0}(y) \right] \\ &= \mathbb{E} \left[\sum_{x \in \mathcal{X}} \chi_{Q_0}(x) \right] + \mathbb{E} \left[\sum_{x, y \in \mathcal{X}, x \neq y} \chi_{Q_0}(x) \chi_{Q_0}(y) \right] \\ &= \int_{Q_0} K_B(x, x) dx + \iint \chi_{Q_0}(x) \chi_{Q_0}(y) \begin{vmatrix} K_B(x, x) & K_B(x, y) \\ K_B(y, x) & K_B(y, y) \end{vmatrix} dx dy \\ &= \lambda^d K_B(0, 0) + \lambda^{2d} K_B(0, 0)^2 - \iint_{Q_0 \times Q_0} |K_B(x, y)|^2 dx dy, \end{aligned}$$

whence

$$\text{Cov}(N_0, N_0) = \text{Var}(N_0) = \lambda^d K_B(0, 0) - \iint_{Q_0 \times Q_0} |K_B(x, y)|^2 dx dy. \quad (20)$$

Now recall that K_B is an orthogonal projection. Thus we have

$$K_B(0, 0) = K_B(x, x) = \int |K_B(x, y)|^2 dy = \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x, y)|^2 dy. \quad (21)$$

The identities (19), (20) and (21) imply that

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0, N_n) &= \lambda^d K_B(0, 0) - \int_{Q_0} dx \sum_{n \in \mathbb{Z}^d} \int_{Q_n} |K_B(x, y)|^2 dy \\ &= \lambda^d K_B(0, 0) - \lambda^d K_B(0, 0) = 0. \end{aligned}$$

Moreover, the above series converge absolutely. Proposition 4.1 is completely proved. \square

Corollary 4.2. *The spectral density $\omega_{N^{(\lambda)}}$ of the stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}^d}$ is a continuous non-negative function on $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$ and vanishes only at $(0, \dots, 0)$.*

Proof. By Lemma 2.3, the spectral density $\omega_{N^{(\lambda)}}$ of the stochastic process $N^{(\lambda)}$ is given by

$$\omega_{N^{(\lambda)}}(\theta_1, \dots, \theta_d) = \sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \dots + n_d\theta_d)}. \quad (22)$$

By (17), the series in (22) converges uniformly and absolutely on \mathbb{T}^d . It follows that $\omega_{N^{(\lambda)}}$ is a continuous function on \mathbb{T}^d .

Now the equality (18) implies that $\omega_{N^{(\lambda)}}(0, \dots, 0) = 0$. Moreover, for any $\theta = (\theta_1, \dots, \theta_d) \in \mathbb{T}^d \setminus \{(0, \dots, 0)\}$, we have

$$\left| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \dots + n_d\theta_d)} \right| < \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})|.$$

By (19), we have

$$|\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| = -\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) \text{ for any } n \in \mathbb{Z}^d \setminus \{0\}.$$

Note that if $\theta = (\theta_1, \dots, \theta_d) \neq (0, \dots, 0)$, then

$$\begin{aligned} \omega_{N^{(\lambda)}}(\theta_1, \dots, \theta_d) &\geq \text{Cov}(N_0^{(\lambda)}, N_0^{(\lambda)}) - \left| \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) e^{i2\pi(n_1\theta_1 + \dots + n_d\theta_d)} \right| \\ &> \text{Cov}(N_0^{(\lambda)}, N_0^{(\lambda)}) - \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |\text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)})| \\ &= \sum_{n \in \mathbb{Z}^d} \text{Cov}(N_0^{(\lambda)}, N_n^{(\lambda)}) = 0. \end{aligned}$$

This shows that $\omega_{N^{(\lambda)}}$ vanishes only at $(0, \dots, 0)$. \square

4.1 Stationary determinantal point processes on \mathbb{R}

Theorem 4.3. *Assume that $B \subset \mathbb{R}$ satisfies*

$$\sup_{R>0} \left(R \int_{|\xi| \geq R} |\widehat{\chi}_B(\xi)|^2 d\xi \right) < \infty. \quad (23)$$

Then the stationary determinantal point process \mathbb{P}_{K_B} is linearly rigid.

Proof. By definition of linear rigidity, we need to show that for any $\lambda > 0$, the stochastic process $N^{(\lambda)} = (N_n^{(\lambda)})_{n \in \mathbb{Z}}$ is linearly rigid. As in the proof of Proposition 4.1, we denote $N_n^{(\lambda)}$ by N_n . By Theorem 3.1, it suffices to show that

$$\sup_{N \geq 1} \left(N \sum_{|n| \geq N} |\text{Cov}(N_0, N_n)| \right) < \infty, \quad (24)$$

and

$$\sum_{n \in \mathbb{Z}} \text{Cov}(N_0, N_n) = 0. \quad (25)$$

By Proposition 4.1, the identity (25) holds in the general case. It remains to prove (24).

By (19), we have

$$\begin{aligned} \sup_{N \geq 1} \left(N \sum_{|n| \geq N} |\text{Cov}(N_0, N_n)| \right) &= \sup_{N \geq 1} N \int_{x \in Q_0} \int_{y \in \bigcup_{|n| \geq N} Q_n} |\widehat{\chi}_B(x - y)|^2 dx dy \\ &= \sup_{N \geq 1} N \int_{-\lambda/2}^{\lambda/2} \int_{|y| \geq (N-1/2)\lambda} |\widehat{\chi}_B(x - y)|^2 dx dy \\ &\leq \sup_{N \geq 1} N \int_{-\lambda/2}^{\lambda/2} \int_{|\xi| \geq (N-1)\lambda} |\widehat{\chi}_B(\xi)|^2 dx dy = \sup_{N \geq 1} \lambda N \int_{|\xi| \geq (N-1)\lambda} |\widehat{\chi}_B(\xi)|^2 d\xi < \infty, \end{aligned}$$

where in the last inequality, we used our assumption (23). Theorem 4.3 is proved completely. \square

Remark 4.1. When B is a finite union of finite intervals on the real line, the rigidity of the stationary determinantal point process \mathbb{P}_{K_B} is due to Ghosh [5].

4.2 Tensor product of sine kernels

In higher dimension, the situation becomes quite different. Let

$$S = I \times I = (-1/2, 1/2) \times (-1/2, 1/2) \subset \mathbb{R}^2.$$

Then the associate kernel K_S has a tensor form: $K_S = K_{\text{sine}} \otimes K_{\text{sine}}$, that is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in \mathbb{R}^2 , we have

$$K_S(x, y) = K_{\text{sine}}(x_1, y_1) K_{\text{sine}}(x_2, y_2) = \frac{\sin(\pi(x_1 - y_1))}{\pi(x_1 - y_1)} \frac{\sin(\pi(x_2 - y_2))}{\pi(x_2 - y_2)}.$$

Proposition 4.4. *The determinantal point process \mathbb{P}_{K_S} is not linearly rigid. More precisely, let $N^{(1)} = (N_n^{(1)})_{n \in \mathbb{Z}^2}$ be the stationary stochastic process given as in Definition 1.2, then*

$$N_0^{(1)} \notin \check{H}_0(N^{(1)}).$$

To prove the above result, we need to introduce some extra notation. First, we define the multiple Zygmund class Λ_* as follows. A continuous function $\varphi(x, y)$ periodic in each variable with period 1 is said to be in the multiple Zygmund class $\Lambda_*(1, 1)$ if for the double difference difference operator $\Delta_{2,2}$ of second order in each variable, applied to φ , there exists a constant $C > 0$, such that for all $x = (x_1, x_2) \in (-1/2, 1/2) \times (-1/2, 1/2)$ and $h_1, h_2 > 0$, we have

$$|\Delta_{2,2}\varphi(x_1, x_2; h_1, h_2)| \leq Ch_1h_2, \quad (26)$$

where

$$\begin{aligned} \Delta_{2,2}\varphi(x_1, x_2; h_1, h_2) &:= \varphi(x_1 + h_1, x_2 + h_2) + \varphi(x_1 - h_1, x_2 + h_2) \\ &\quad + \varphi(x_1 + h_1, x_2 - h_2) + \varphi(x_1 - h_1, x_2 - h_2) - 2\varphi(x_1 + h_1, x_2) \\ &\quad - 2\varphi(x_1 - h_1, x_2) - 2\varphi(x_1, x_2 + h_2) - 2\varphi(x_1, x_2 - h_2) + 4\varphi(x_1, x_2). \end{aligned}$$

The following result is due to Fülöp and Móricz [4, Thm 2.1 and Rem. 2.3]

Theorem 4.5 (Fülöp-Móricz). *If $\{c_{jk}\}_{j,k \in \mathbb{Z}} \in \mathbb{C}$ is such that*

$$\sup_{N \geq 1, M \geq 1} \left(MN \sum_{|j| \geq N, |k| \geq M} |c_{jk}| \right) < \infty, \quad (27)$$

then the function

$$\varphi(\theta_1, \theta_2) = \sum_{j,k \in \mathbb{Z}} c_{jk} e^{i2\pi(j\theta_1 + k\theta_2)}$$

is in the Zygmund class $\Lambda_(1, 1)$.*

Let us turn to the study of the density function $\omega_{N^{(1)}}$.

Lemma 4.6. *There exists $c > 0$, such that for any $\theta_1, \theta_2 \in [-1/2, 1/2]$, we have*

$$\omega_{N^{(1)}}(\theta_1, \theta_2) \geq c(|\theta_1| + |\theta_2|). \quad (28)$$

Proof. To make notation lighter, in this proof we simply write ω for $\omega_{N^{(1)}}$.

For any $n = (n_1, n_2) \in \mathbb{Z}^2$, let us denote $S_n = S \times (n + S)$ where

$$n + S := (-1/2 + n_1, 1/2 + n_1) \times (-1/2 + n_2, 1/2 + n_2).$$

By the same argument as in the proof of Proposition 4.1, we obtain that for any $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus \{0\}$,

$$\widehat{\omega}(n) = - \int_{S_n} |K_S(x, y)|^2 dx dy, \quad (29)$$

and

$$\widehat{\omega}(0) = K_S(0, 0) - \int_{S_0} |K_S(x, y)|^2 dx dy.$$

The following properties can be easily checked.

- $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0$.
- $\widehat{\omega}(\varepsilon_1 n_1, \varepsilon_2 n_2) = \widehat{\omega}(n_1, n_2)$, where $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$.
- there exist $c, C > 0$, such that

$$\frac{c}{(1 + n_1^2)(1 + n_2^2)} \leq |\widehat{\omega}(n_1, n_2)| \leq \frac{C}{(1 + n_1^2)(1 + n_2^2)}.$$

For instance, $\sum_{n \in \mathbb{Z}^2} \widehat{\omega}(n) = 0$ follows from Proposition 4.1. These properties combined with Theorem 4.5 yield that

- $\omega(0, 0) = 0$.
- $\omega(\varepsilon_1 \theta_1, \varepsilon_2 \theta_2) = \omega(\theta_1, \theta_2)$ for any $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$ and $\theta_1, \theta_2 \in (-1/2, 1/2)$.
- the function $\omega(\theta_1, \theta_2)$ is in the multiple Zygmund class $\Lambda_*(1, 1)$.

Hence there exists $C > 0$, such that

$$|\omega(\theta_1, \theta_2) - \omega(\theta_1, 0) - \omega(0, \theta_2)| \leq C|\theta_1 \theta_2|. \quad (30)$$

Lemma 4.7. *There exists $c_1 > 0$, such that*

$$\omega(\theta_1, 0) \geq c_1|\theta_1| \text{ and } \omega(0, \theta_2) \geq c_1|\theta_2|. \quad (31)$$

Let us postpone the proof of Lemma 4.7 and proceed to the proof of Lemma 4.6. The inequalities (30) and (31) imply that

$$\omega(\theta_1, \theta_2) \geq c_1(|\theta_1| + |\theta_2|) - C|\theta_1 \theta_2|.$$

Now if $|\theta_1|$ is small enough such that $2C|\theta_1| \leq c_1$, then we have

$$\omega(\theta_1, \theta_2) \geq \frac{c_1}{2}(|\theta_1| + |\theta_2|).$$

If $2C|\theta_1| \geq c_1$, by Corollary 4.2, the function $\omega(\theta_1, \theta_2)$ is continuous on $[-\frac{1}{2}, \frac{1}{2}]^2$ and vanishes only at $(0, 0)$. Consequently,

$$\inf_{|\theta_1| \geq c_1/2C} \omega(\theta_1, \theta_2) = c_2 > 0.$$

It follows, by using the elementary fact that $|\theta_1| + |\theta_2| \leq 1$, that

$$\inf_{|\theta_1| \geq c_1/2C} \omega(\theta_1, \theta_2) = c_2 \geq \frac{c_2}{2}(|\theta_1| + |\theta_2|).$$

Taking $c = \min(\frac{c_1}{2}, \frac{c_1}{2})$, we get the desired inequality (28). \square

Now let us turn to the proof of Lemma 4.7.

Proof of Lemma 4.7. By symmetry, it suffices to prove that there exists $c > 0$, such that $\omega(\theta_1, 0) \geq c|\theta_1|$. To this end, let us denote $\omega_1(\theta_1) := \omega(\theta_1, 0)$. Then $\omega_1(0) = 0$ and there exists $c > 0$ such that if $k \neq 0$, then

$$\widehat{\omega}_1(k) < 0 \text{ and } |\widehat{\omega}_1(k)| \geq c/(1 + k^2).$$

Indeed, we have

$$\omega_1(\theta_1) = \sum_{k \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \widehat{\omega}(k, n_2) e^{i2\pi k \theta_1}.$$

If $k \neq 0$, then by (29), we have $\widehat{\omega}(k, n_2) < 0$ and hence

$$|\widehat{\omega}_1(k)| = \sum_{n_2 \in \mathbb{Z}} |\widehat{\omega}(k, n_2)| \geq \sum_{n_2 \in \mathbb{Z}} \frac{c}{(1 + n_2^2)(1 + k^2)} \geq \frac{c'}{1 + k^2}. \quad (32)$$

We claim that $\omega_1(0) = 0$. Indeed, by definition, we have

$$\omega_1(0) = \sum_{k \in \mathbb{Z}} \sum_{n_2 \in \mathbb{Z}} \widehat{\omega}(k, n_2) = \omega(0, 0) = 0,$$

where in the last equality, we used Corollary 4.2 that claims $\omega(0, 0) = 0$. Now we have

$$\sum_{k \in \mathbb{Z}} \widehat{\omega}_1(k) = \omega_1(0) = 0.$$

It follows that

$$\begin{aligned} \omega_1(\theta_1) &= \sum_{k \in \mathbb{Z}} \widehat{\omega}_1(k) e^{i2\pi k \theta_1} = \sum_{k \in \mathbb{Z}} \widehat{\omega}_1(k) \left(\frac{e^{i2\pi k \theta_1} + e^{-i2\pi k \theta_1}}{2} - 1 \right) \\ &= \sum_{k \in \mathbb{Z}, k \neq 0} -\widehat{\omega}_1(k) (1 - \cos(2\pi k \theta_1)) = \sum_{k \in \mathbb{Z}, k \neq 0} |\widehat{\omega}_1(k)| (1 - \cos(2\pi k \theta_1)). \end{aligned}$$

Since $|\widehat{\omega}_1(k)|(1 - \cos(2\pi k\theta_1))$ is non-negative for any $k \in \mathbb{Z}$, we have

$$\omega_1(\theta_1) \geq \sum_{j=1}^{\infty} |\widehat{\omega}_1(2j-1)|(1 - \cos(2\pi(2j-1)\theta_1)).$$

The inequality (32) implies that there exists $c'' > 0$, such that $|\widehat{\omega}_1(2j-1)| \geq \frac{c''}{(2j-1)^2}$, hence we obtain that

$$\omega_1(\theta_1) \geq c'' \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} (1 - \cos(2\pi(2j-1)\theta_1)).$$

Combining with the Fourier series of the absolutely value function (the Fourier coefficient of the absolute value function on $(-\frac{1}{2}, \frac{1}{2})$ can be computed explicitly):

$$|\alpha| = \frac{1}{4} - \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{\cos(2\pi(2j-1)\alpha)}{(2j-1)^2}, \text{ for } \alpha \in (-1/2, 1/2);$$

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8} \text{ (take } \alpha = 0 \text{ in the above series),}$$

we obtain that

$$\begin{aligned} \omega_1(\theta_1) &\geq c'' \left(\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} - \sum_{j=1}^{\infty} \frac{\cos(2\pi(2j-1)\theta_1)}{(2j-1)^2} \right) \\ &= c'' \left(\frac{\pi^2}{8} + \frac{\pi^2}{2} \left(|\theta_1| - \frac{1}{4} \right) \right) = c'' \frac{\pi^2}{2} |\theta_1|. \end{aligned}$$

The proof of Lemma 4.7 is complete. \square

Proof of Proposition 4.4. By Lemma 2.1, it suffices to show that

$$\int_{\mathbb{T}^2} \omega_{N(1)}^{-1} dm < \infty. \quad (33)$$

By Lemma 4.6, the inequality (33) follows from the following elementary inequality

$$\int_{|\theta_1| < 1/2, |\theta_2| < 1/2} \frac{1}{|\theta_1| + |\theta_2|} d\theta_1 d\theta_2 < \infty.$$

\square

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